Integrals of the Motion for Optimal Trajectories in Atmospheric Flight

Nguyen X. Vinh*

The University of Michigan, Ann Arbor, Mich.

The problem of optimizing the flight trajectory of a rocket vehicle moving in a resisting medium, in a general gravitational force field, is considered. Optimal laws for the modulation of the aerodynamic and propulsive forces are formulated in terms of the primer vector, the vector adjoint to the velocity. Relations for flight along an intermediate-thrust arc and integrals of the motion for several cases of practical interest are derived.

Nomenclature

a, a_i = constants of integration
A_i = Cartesian components of $\vec{A}$
$\vec{A}$ = aerodynamic force
b, b_i = constants of integration
c = constant exhaust velocity
C_D = drag coefficient
C_{D_0} = zero-lift drag coefficient
C_L = lift coefficient
D = drag
$g, \vec{g}$ = acceleration of the gravity, vector
H = Hamiltonian constant
k = induced-drag coefficient
K = switching function
L = lift
m = mass
p = magnitude of $\vec{p}$
p_i = Cartesian components of $\vec{p}$
$p_m, p_v,$ = adjoint components, associated to the respective variables
p_γ, p_ψ
$\vec{p}$ = adjoint vector, associated to $\vec{V}$
q_i = Cartesian component of $\vec{q}$
$\vec{q}$ = adjoint vector, associated to $\vec{r}$
$\vec{r}$ = position vector
S = reference area
t = time
T = thrust magnitude
T_{max} = maximum thrust magnitude
v_i = Cartesian components of $\vec{V}$
V = magnitude of $\vec{V}$
$\vec{V}$ = velocity vector
x_i = Cartesian components of $\vec{r}$
β = constant height scale in exponential atmosphere
γ = flight path angle
$\nabla(\ )$ = gradient vector
ε = angle between $\vec{V}$ and $\vec{p}$
λ = lift control, Eq. (4)
ϕ = bank angle
ψ = heading angle
ρ = atmospheric mass density
ρ_0 = atmospheric mass density at reference level

Introduction

THE problem of determining optimal trajectories for a rocket powered lifting vehicle, flying inside the atmosphere of a planet, has received considerable attention in recent years. General control laws for the lift and the bank angle, and for the thrusting program have been obtained in terms of the primer vector, the adjoint vector associated to the velocity vector.[1,2] In this paper, we modify the results in Ref. 1 to apply to the case when the thrust direction is constrained to align with the velocity vector. This is the case when the engine is mounted fixed with respect to the vehicle and the flight is accomplished in the dense layer of the atmosphere and at small angle of attack to validate the assumption.[3] From the general theory, solutions for optimal maneuvers for flight in a uniform gravitational field will be obtained by canonical transformations.

The Optimal Controls

Consider the motion of a powered, lifting vehicle, in a general gravitational force field. At the time t, the state of the vehicle is defined by its position vector $\vec{r}(t)$, velocity $\vec{V}(t)$ and instantaneous mass $m(t)$. The flight is controlled by a thrusting force $\vec{T}$ and the aerodynamic force $\vec{A}$ (Fig. 1). It is assumed that $\vec{T}$ is aligned with the velocity, and its magnitude is bounded by

$$0 \leq T \leq T_{max} \qquad (1)$$

Furthermore, we assume that the vehicle has a plane of symmetry, both the thrust and the aerodynamic force are applied at the center of mass, and in coordinated flight, the aerodynamic force and the velocity are contained in that plane of symmetry. It is customary to decompose $\vec{A}$ into a drag force $\vec{D}$, always opposite to $\vec{V}$, and a lift force $\vec{L}$ orthogonal to it. We shall use the usual assumption

$$L = (\tfrac{1}{2})\rho S V^2 C_L ; \qquad D = (\tfrac{1}{2})\rho S V^2 C_D \qquad (2)$$

where, in general, the atmospheric mass density is function of $\vec{r}$. In hypersonic flight, the lift coefficient C_L, and the drag coefficient C_D are assumed independent of the Mach number
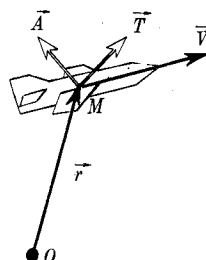

Fig. 1 State and control variables.

and the Reynolds number. For simplicity, we shall assume a parabolic drag polar, defined by the relation

$$C_D = C_{D_0} + kC_L^2 \qquad (3)$$

where C_{D_0} and k are constants. It is convenient to define a lift control parameter λ such that

$$\lambda = (k/C_{D_0})^{1/2}C_L \qquad (4)$$

Then, when $\lambda = 1$, the flight is at maximum lift-to-drag ratio.

The motion of the vehicle, flying in a general gravitational force field, and subject to aerodynamic force and thrusting force, is governed by the equations

$$d\vec{r}/dt = \vec{V}$$
$$d\vec{V}/dt = (T/m)(\vec{V}/V) + (\vec{A}/m) + \vec{g}(\vec{r}, t) \qquad (5)$$
$$dm/dt = -(T/c)$$

where the exhaust velocity c of the gas ejected from the engine is assumed constant. The optimal transfer problem is defined as follows.

At the initial time, $t = 0$, $\vec{r} = \vec{r}_0$, $\vec{V} = \vec{V}_0$, $m = m_0$. The vectors $\vec{r}_0$, $\vec{V}_0$, and the scalar m_0 are prescribed. At the final time, $t = t_f$, $\vec{r} = \vec{r}_f$, $\vec{V} = \vec{V}_f$, and $m = m_f$. The problem is to find the time history of $\vec{A}(t)$ and $T(t)$ such that some scalar function of the final state is a minimum.

Using the maximum principle, we introduce the adjoint elements $\vec{q}$, $\vec{p}$ and p_m to form the Hamiltonian

$$H = \vec{q} \cdot \vec{V} + \vec{p} \cdot (\vec{A}/m + \vec{g}) + (T/m)[(\vec{p} \cdot \vec{V})/V - mp_m/c] \qquad (6)$$

where $\vec{q}$, $\vec{p}$, and p_m are defined by the adjoint equations

$$d\vec{q}/dt = -\nabla(\vec{p} \cdot \vec{g}) - m^{-1}\nabla(\vec{p} \cdot \vec{A})$$
$$d\vec{p}/dt = -\vec{q} - 2[(\vec{p} \cdot \vec{A})/mV^2]\vec{V} - (T/mV)[\vec{p} - (\vec{p} \cdot \vec{V})V^{-2}\vec{V}] \qquad (7)$$
$$dp_m/dt = (T/m^2V)(\vec{p} \cdot \vec{V}) + m^{-2}(\vec{p} \cdot \vec{A})$$

The following general results for optimal trajectories have been obtained in Ref. 1.

1) If the direction of the thrust can be taken arbitrarily, whenever the engine is operating, the thrust must be directed along the vector $\vec{p}$, called the primer vector. This is the extension of the classical result obtained by Lawden for transfer in a vacuum.

However, it should be noted that a wide variety of airborne vehicles are not free to alter their thrust direction independently of the angle of attack. Hence, for the type of vehicles considered in this paper, we shall assume that the engine is mounted fixed with respect to the airframe, and the thrust line is taken as the body axis from which we measure the angle of attack. Furthermore, in hypersonic flight, especially during the thrusting phase, the angle of attack is small, and we shall assume that the thrust is always aligned with the velocity as expressed by the second equation (5). For the validity of this assumption, we refer to the classical paper by Bryson and Lele.[3]

2) The thrusting program is governed by the switching function, which, for the case where the thrust is aligned with the velocity, is defined by

$$K = (\vec{p} \cdot \vec{V})/V - mp_m/c \qquad (8)$$

If $K > 0$, we select $T = T_{max}$ (boosting phase); if $K < 0$, we select $T = 0$ (coasting phase); and if $K = 0$, for a finite time interval, we select $T =$ variable (sustaining phase).

3) The optimal lift modulation is such that $C_L = C_{L_{max}}$, or, for variable lift program

$$\tan \varepsilon = \partial C_D/\partial C_L \qquad (9)$$

where ε is the angle between $\vec{V}$ and $\vec{p}$.

4) The optimal bank angle is such that the vectors $\vec{V}$, $\vec{p}$, and $\vec{A}$ are coplanar, that is

$$(\vec{V} \times \vec{p}) \cdot \vec{A} = 0 \qquad (10)$$

5) For optimal flight constantly at maximum lift-to-drag ratio, $\vec{p}$ is orthogonal to $\vec{A}$, that is

$$\vec{p} \cdot \vec{A} = 0 \qquad (11)$$

6) Along a variable-thrust arc, we have

$$(\vec{p} \cdot \vec{V})/V - mp_m/c = 0 \qquad (12)$$

By taking the derivative of this equation, and using Eqs. (5–7) together with Eq. (12) and rearranging, we have

$$H + (V/mc)(\vec{p} \cdot \vec{A}) + (\vec{p} \cdot \vec{V})[(\vec{V} \cdot \vec{A})/mV^2 + (\vec{V} \cdot \vec{g})/V^2] = 2(\vec{p} \cdot \vec{g}) \qquad (13)$$

The variable thrust magnitude control will be obtained upon taking the derivative of this equation.

These results are valid for a general gravitational force field. In particular, when $\vec{g}$ is time invariant, the Hamiltonian given by Eq. (6) is constant.

Application to the Case of a Uniform Gravitational Field

Consider the case where $\vec{g}$ is a constant vector. In Cartesian rectangular coordinates, let

$$\vec{r} = (x_1, x_2, x_3) \quad \text{with} \quad x_3 \quad \text{along} \quad \vec{g}$$
$$\vec{V} = (v_1, v_2, v_3)$$
$$\vec{q} = (q_1, q_2, q_3)$$
$$\vec{p} = (p_1, p_2, p_3)$$
$$\vec{A} = (A_1, A_2, A_3)$$

We shall use the transformation

$$v_1 = V \cos \gamma \cos \psi; \quad v_2 = V \cos \gamma \sin \psi; \quad v_3 = V \sin \gamma \qquad (14)$$

It is seen that γ is the flight-path angle, measured from the horizontal plane, and ψ the heading, measured from a reference vertical plane. We shall consider an exponential atmosphere of the form

$$\rho = \rho_0 \exp(-\beta x_3) \qquad (15)$$

where β is the constant height scale. For the transformation (14) to be canonical, while conserving the same Hamiltonian

$$p_1 dv_1 + p_2 dv_2 + p_3 dv_3 = p_v dV + p_\gamma d\gamma + p_\psi d\psi$$

This gives the linear transformation for the adjoint components of the primer vector

$$\begin{bmatrix} \cos \gamma \cos \psi & \cos \gamma \sin \psi & \sin \gamma \\ -\sin \gamma \cos \psi & -\sin \gamma \sin \psi & \cos \gamma \\ -\cos \gamma \sin \psi & \cos \gamma \cos \psi & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} p_v \\ p_\gamma/V \\ p_\psi/V \end{bmatrix} \qquad (16)$$

Reversing the matrix equation

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \cos \gamma \cos \psi & -\sin \gamma \cos \psi & -\sin \psi/\cos \gamma \\ \cos \gamma \sin \psi & -\sin \gamma \sin \psi & \cos \psi/\cos \gamma \\ \sin \gamma & \cos \gamma & 0 \end{bmatrix} \begin{bmatrix} p_v \\ p_\gamma/V \\ p_\psi/V \end{bmatrix} \qquad (17)$$

In particular, we can verify the following relations

$$p^2 = p_1^2 + p_2^2 + p_3^2 = p_v^2 + p_\gamma^2/V^2 + p_\psi^2/V^2 \cos^2 \gamma \qquad (18)$$

and

$$\vec{p} \cdot \vec{V} = p_1 v_1 + p_2 v_2 + p_3 v_3 = Vp_v \qquad (19)$$

For the transformation of the aerodynamic forces, we define the bank angle ϕ as the angle between the plane of symmetry of the airplane and the vertical plane passing through the velocity $\vec{V}$ (Fig. 2). Then, from the geometry of the figure

$$(A_1 \cos \psi + A_2 \sin \psi) \cos \gamma + A_3 \sin \gamma = -D$$
$$(A_1 \cos \psi + A_2 \sin \psi) \sin \gamma - A_3 \cos \gamma = -L \cos \phi \qquad (20)$$
$$A_1 \sin \psi - A_2 \cos \psi = -L \sin \phi$$

Therefore, we have for the Cartesian components of the aerodynamic force

$$A_1 = -D \cos \psi \cos \gamma - L(\cos \phi \cos \psi \sin \gamma + \sin \phi \sin \psi)$$
$$A_2 = -D \sin \psi \cos \gamma - L(\cos \phi \sin \psi \sin \gamma - \sin \phi \cos \psi) \qquad (21)$$
$$A_3 = -D \sin \gamma + L \cos \phi \cos \gamma$$

The aerodynamic optimal controls can be obtained directly from the general theory. Using formula (9), with relation (3), we have for the optimal variable lift control

$$\tan \varepsilon = 2kC_L \qquad (22)$$

For the bank angle, by expanding the determinant (10), using Eqs. (14) and (17) for the components of $\vec{V}$ and $\vec{p}$, we have
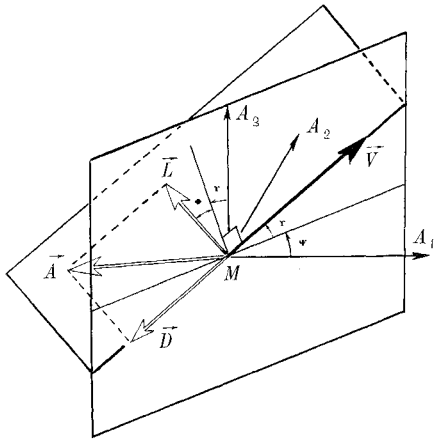
Fig. 2   Aerodynamic forces transformation.

$$p_\gamma(A_1 \sin\psi - A_2 \cos\psi) -$$
$$(p_\psi/\cos\gamma)[(A_1 \cos\psi + A_2 \sin\psi)\sin\gamma - A_3 \cos\gamma] = 0$$

By relations (20), we have for the optimal variable bank angle

$$\tan\phi = p_\psi/p_\gamma \cos\gamma \qquad (23)$$

We can rewrite relation (22) by observing that

$$\vec{p}\cdot\vec{V} = pV\cos\varepsilon = Vp_v$$

therefore

$$p^2 = p_v{}^2/\cos^2\varepsilon \qquad (24)$$

On the other hand, from Eq. (18), and using the optimal law (23)

$$p^2 = p_v{}^2 + p_\gamma{}^2/V^2 + p_\gamma{}^2 \tan^2\phi/V^2$$

or

$$p^2 = p_v{}^2 + p_\gamma{}^2/V^2 \cos^2\phi \qquad (25)$$

By eliminating p^2 between Eqs. (24) and (25), we have

$$\tan\varepsilon = p_\gamma/Vp_v \cos\phi \qquad (26)$$

The optimal lift control, expressed in terms of the adjoint variables, is then

$$2kC_L = p_\gamma/Vp_v \cos\phi \qquad (27)$$

We see that the aerodynamic controls are governed by the adjoint components p_v, p_γ, and p_ψ of the primer vector.

The Integrals of the Motion

A number of the integrals of the motion, mostly associated with a flight at maximum lift-to-drag ratio, has been displayed in Ref. 1. Here, we shall derive them directly and furthermore, obtain some additional integrals.

The equations of the motion, written in components form, and using the velocity transformation (14), are

$$dx_1/dt = V\cos\gamma\cos\psi; \quad dx_2/dt = V\cos\gamma\sin\psi$$
$$dx_3/dt = V\sin\gamma; \quad dV/dt = (T-D)/m - g\sin\gamma \qquad (28)$$
$$d\gamma/dt = L\cos\phi/mV - g\cos\gamma/V$$
$$d\psi/dt = L\sin\phi/mV\cos\gamma, \quad dm/dt = -T/c$$

The Hamiltonian equation (6) becomes, with the new variables, in a uniform gravitational field

$$H = (q_1 \cos\psi + q_2 \sin\psi)V\cos\gamma + q_3 V\sin\gamma -$$
$$p_v(D/m + g\sin\gamma) + p_\gamma(L\cos\phi/mV - g\cos\gamma/V) +$$
$$p_\psi L\sin\phi/mV\cos\gamma + (T/m)(p_v - mp_m/c) \qquad (29)$$

Along an optimal trajectory, with variable lift and bank controls as given by Eqs. (23) and (27), we have

$$H = (q_1 \cos\psi + q_2 \sin\psi)V\cos\gamma + q_3 V\sin\gamma -$$
$$g(p_v \sin\gamma + p_\gamma \cos\gamma/V) + (kC_L{}^2 - C_{D_0})(\rho SV^2 p_v/2m) +$$
$$(T/m)(p_v - mp_m/c) \qquad (30)$$

We notice that, if the flight is at maximum lift-to-drag ratio, $kC_L{}^2 - C_{D_0} = 0$, and the constant Hamiltonian is free of aerodynamic components. This result has been shown to be valid for general gravitational force field in Ref. 1. The switching function is seen to be

$$K = p_v - mp_m/c \qquad (31)$$

The adjoint equations, with optimal variable lift and bank controls are

$$dq_1/dt = 0$$
$$dq_2/dt = 0$$
$$dq_3/dt = (kC_L{}^2 - C_{D_0})(\beta\rho SV^2 p_v/2m)$$
$$dp_v/dt = -(q_1 \cos\psi + q_2 \sin\psi)\cos\gamma - q_3 \sin\gamma -$$
$$\qquad gp_\gamma \cos\gamma/V^2 + \rho SVC_{D_0} p_v/m \qquad (32)$$
$$dp_\gamma/dt = (q_1 \cos\psi + q_2 \sin\psi)V\sin\gamma - q_3 V\cos\gamma +$$
$$\qquad g(p_v \cos\gamma - p_\gamma \sin\gamma/V) - p_\psi L\sin\phi \sin\gamma/m V\cos^2\gamma$$
$$dp_\psi/dt = (q_1 \sin\psi - q_2 \cos\psi)V\cos\gamma$$
$$dp_m/dt = (kC_L{}^2 - C_{D_0})(\rho SV^2 p_v/2m^2) + Tp_v/m^2$$

We first have the integrals

$$q_1 = a_1 = \text{const}; \qquad q_2 = a_2 = \text{const} \qquad (33)$$

If the longitudinal range and the lateral range are free, these constants are zero. It is seen that q_3 is also constant if the flight is at maximum lift-to-drag ratio. Next, using Eq. (33) we have

$$d(q_1 x_2 - q_2 x_1)/dt = dp_\psi/dt$$

Hence

$$q_1 x_2 - q_2 x_1 = p_\psi - a_6 \qquad (34)$$

where a_6 is a constant of integration. For free longitudinal and lateral ranges

$$p_\psi = a_6 \qquad (35)$$

If the final heading is not prescribed, this constant is zero, and from Eq. (23), $\phi = 0$. The flight is performed in a vertical plane containing the initial velocity.

Now, consider

$$d(mp_m)/dt = (kC_L{}^2 - C_{D_0})(\rho SV^2 p_v/2m) + (T/m)(p_v - mp_m/c) \qquad (36)$$

First, along a coasting arc, $T = 0$, or along a sustaining arc, $K = 0$, and if the flight is at maximum lift-to-drag ratio

$$mp_m = \text{const} \qquad (37)$$

For flight at variable lift coefficient, and along a coasting or sustaining arc, by eliminating the time between Eq. (36) and the equation for q_3, we have

$$dq_3 = \beta d(mp_m)$$

By integrating, we have

$$q_3 = \beta mp_m + a_3 \qquad (38)$$

where a_3 is a new constant of integration.

Consider the time derivative of Vp_v

$$d(Vp_v)/dt = -H - 2g(p_v \sin\gamma + p_\gamma \cos\gamma/V) + (T/m)(2p_v - mp_m/c) \qquad (39)$$

Along a coasting arc, $T = 0$, and using Eggers assumption by neglecting the gravity compared to aerodynamic force,[4] an assumption generally valid for a skip or pull-up maneuver, we have, upon integration

$$Vp_v = -Ht + a_4 \qquad (40)$$

where a_4 is a constant of integration. In particular, when the final time is free, $H \equiv 0$, and

$$Vp_v = a_4 \qquad (41)$$

We write the Hamiltonian for the case of free longitudinal and lateral ranges, and free time, along a coasting or a sustaining arc. Relation (30) becomes

$$(kC_L{}^2 - C_{D_0})(\rho SV^2 p_v/2m) =$$
$$-q_3 V\sin\gamma + g(p_v \sin\gamma + p_\gamma \cos\gamma/V) \qquad (42)$$

With this integral, we can delete the differential equation for p_γ. Using Eq. (42) we rewrite the equation for q_3

$$dq_3/dt = -\beta q_3 V \sin\gamma + \beta g(p_v \sin\gamma + p_\gamma \cos\gamma/V)$$

The equation can be integrated if skip trajectory assumption is used. We have

$$dq_3/dt = -\beta q_3 V \sin\gamma = -\beta q_3 \, dx_3/dt$$

Upon integrating

$$q_3 = \text{const} \times \exp(-\beta x_3) = \text{const} \times \rho \qquad (43)$$

By the change of variable from x_3 to ρ, using the exponential law (15), a canonical transformation requires

$$q_3 \, dx_3 = q_\rho \, d\rho = -\beta\rho q_\rho \, dx_3$$

Hence

$$q_3 = -\beta\rho q_\rho$$

and therefore

$$q_\rho = \text{const} \qquad (44)$$

This shows the advantage of using the atmospheric mass density as the altitude variable.

The Eggers assumption, in the free range and free time case, gives explicit laws for the modulation of the lift and the bank.

First, along a coasting arc, we rewrite Eq. (43)

$$q_3 = b_3 \rho \qquad (45)$$

where b_3 is a constant. The Hamiltonian relation (42) gives

$$V p_v = 2b_3 m \sin\gamma / S(C_{D_0} - kC_L^2)$$

Since in this case $V p_v = a_4$, we have

$$kC_L^2/C_{D_0} = 1 - 2b_3 m \sin\gamma / S a_4 C_{D_0}$$

That is, by observing that m is constant along a coasting arc

$$\lambda^2 = 1 + a\sin\gamma \qquad (46)$$

where a is a constant. We have the classical result, first obtained by Contensou,[5] and later extended to three-dimensional case by Griffin and Vinh,[6] and Speyer and Womble.[7] We now rewrite the law for the bank angle

$$\tan\phi = p_\psi/p_\gamma \cos\gamma = a_6/2ka_4 C_L \cos\phi \cos\gamma$$

or

$$\sin\phi = b/\lambda \cos\gamma \qquad (47)$$

where b is a constant. The constants a and b in the optimal lift and bank laws are determined by specifying terminal conditions at the ends of a coasting arc.

Next, along a sustaining arc, we have the relations (12) and (13) which for a constant gravitational field, become

$$p_v - m p_m/c = 0 \qquad (48)$$

and

$$(\rho S V^2 p_v/2m)[(V/c+1)C_{D_0} - (V/c-1)kC_L^2] =$$
$$H + g(2p_\gamma \cos\gamma/V + p_v \sin\gamma) \qquad (49)$$

For a free-time problem, $H = 0$, and neglecting gravity, we have the following simple law for optimal variable lift control along a sustaining arc

$$\lambda^2 = (V/c+1)/(V/c-1) \qquad (50)$$

An interesting fact is that variable lift control for sustaining flight is only optimal for high velocity, $V > c$, and for vehicle with high lift performance, $\lambda > 1$.

Finally, for the boosting arc, by eliminating the time between the equations for V and m, we have

$$dV/dm = -c/m + (c/T_{\max})(D/m + g\sin\gamma)$$

With $T_{\max} \to \infty$ (impulsive approximation), the equation can be integrated to give

$$m = m^* \exp(-V/c) \qquad (51)$$

where m^* is a constant mass.

Conclusions

General control laws for the modulation of the lift and the bank angle, and for the thrusting program along the optimal trajectory of a rocket powered, lifting vehicle, flying inside the atmosphere of a planet, with the thrust aligned along the velocity vector, have been obtained in terms of the primer vector, the adjoint vector associated to the velocity vector. From the general theory, solutions for optimal maneuvers for flight in a uniform gravitational field are obtained by canonical transformations. The problem is completely solved for the free range and free time case, using Eggers assumption for skip trajectory. It should be noted that this last stringent condition can be removed by using Loh's second-order theory,[8] as applied by Speyer and Womble for free flight trajectory.[7] Loh's second-order theory, which also includes the curvature of the flight path, has also been applied successfully to the case of thrusting flight by Griffin and Vinh.[9]

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